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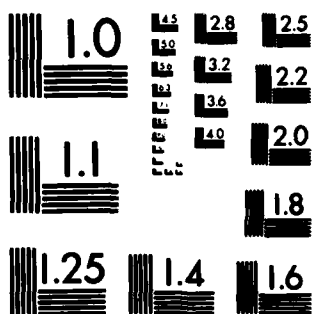
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A NONLINEAR EIGENVALUE PROBLEM
MODELLING THE AVALANCHE EFFECT
IN SEMICONDUCTOR DIODES

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AVALANCHE EFFECT IN SEMICONDUCTOR DIODES

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ABSTRACT

This paper is concerned with the analysis of the solution set of the two-point boundary value problem modelling the avalanche effect in semiconductor diodes for negative applied voltage. - This effect is represented by a large increase of the absolute value of the current starting at a certain reverse basis. We interpret the avalanche-model as a nonlinear eigenvalue problem (with the current as eigenparameter) and show (using a priori estimates and a well known theorem on the structure of solution sets of nonlinear eigenvalue problems for compact operators) that there exists an unbounded continuum of solutions which contains a solution corresponding to every negative voltage. Therefore, the solution branch does not "break down" at a certain threshold voltage (as expected on physical grounds). We discuss the current-voltage characteristic and prove that the absolute value of the current increases at most (and at least) exponentially in the avalanche case as the voltage decreases to minus infinity.

AMS (MOS) Subject Classifications: 34B15, 58C40, 78A25

Key Words: Nonlinear eigenvalue problems, unbounded solution continua,
two-point boundary-value problems, impact ionization,
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Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

In this paper we investigate the mathematical model equations for impact ionization in a semiconductor diode. This effect (also called avalanche generation) is characterized by a "sudden" increase of the current flowing through the device starting at a certain negative voltage. Physically, the diode "breaks down" shortly after the onset of avalanche generation. Therefore, it was conjectured that there is a threshold voltage beyond which no solutions of the avalanche model exists. We show that this conjecture is false; more precisely a continuous branch of solution along which every negative voltage and every negative bias is assumed (at least once) exists. Mathematically, the avalanche-effect only becomes apparent through an exponential increase of the absolute value of the current starting at a certain negative voltage.

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A NONLINEAR EIGENVALUE PROBLEM MODELLING THE
AVALANCHE EFFECT IN SEMICONDUCTOR DIODES

Peter A. Markovich*

1. INTRODUCTION

We investigate the (one-dimensional) boundary-value problem which describes the performance of a semiconductor diode in the case of avalanche generation. The physical situation is as follows. A semiconductor is doped with donor atoms on the right side (n-side) and with acceptor atoms on the left side (p-side) and a bias is applied to the Ohmic contacts (see Figure 1).

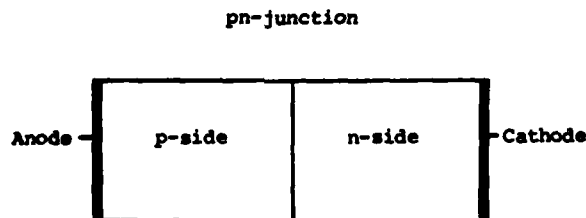


Figure 1: Diode

For simplicity we assume that the pn-junction is in the middle of the device, that the doping profile (that is the difference of the concentrations of donors and acceptors) is constant in the n-side as well as in the p-side and odd about the pn-junction.

A well-known phenomenon is the 'breakdown' of the diode due to impact ionization (avalanche generation, see Sze (1981)) under sufficiently large negative bias. This 'breakdown' is based on a 'sudden' increase of the current (as a function of the applied bias).

To study the current-voltage (J-V) characteristic of the device we investigate the basic semiconductor device equations describing potential and carrier distributions in the

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diode (see Van Roosbroeck (1950), Sze (1981));

$$\left. \begin{aligned} (1.1) \quad \lambda^2 \psi'' &= n - p - D && \text{Poisson's equation} \\ (1.2) \quad n' &= n\psi' + J_n && \text{electron continuity equation} \\ (1.3) \quad p' &= p\psi' - J_p && \text{hole continuity equation} \end{aligned} \right\} -1 \leq x \leq 1$$

ψ denotes the electrostatic potential, ψ' is the electric field, $n(p)$ the electron (hole) density, $J_n(J_p)$ the electron (hole) current density and D the doping profile. The equations (1.1)-(1.3) are already in dimensionless form, the doping profile is scaled to maximally one and the independent variable x to $[-1, 1]$. In our symmetric and piecewise constant case

$$(1.4) \quad D = \begin{cases} 1, & 0 \leq x \leq 1 \quad (\text{n-side}) \\ -1, & -1 \leq x < 0 \quad (\text{p-side}) \end{cases}$$

holds. The pn-junction is at $x = 0$. $\lambda^2 (\ll 1)$ is a scaling parameter.

Generally the current relations are given by

$$(1.5) \quad (a) \quad J_n' = R, \quad (b) \quad J_p' = -R$$

where the recombination-generation term R is a nonlinear function of n , p , J_n , J_p and ψ' (the electric field). We assume that R is given by the avalanche-generation term (see Sze (1981), Schütz (1982));

$$(1.6) \quad R = R(J_n, J_p, \psi') = -\alpha(\psi')(|J_n| + |J_p|)$$

where $\alpha > 0$ is the electron-hole ionization rate. α is strongly field-dependent.

Commonly used α 's are $\alpha(\tau) = \gamma e^{-\frac{\sigma}{|\tau|}}$, $\alpha(\tau) = \gamma |\tau| e^{-\frac{\sigma}{|\tau|}}$, $\gamma, \sigma > 0$. For simplicity we assume that $\alpha : C([0, 1]) \rightarrow [0, \gamma]$, $\gamma > 0$ is the nonnegative functional

$$(1.7) \quad \alpha(f) = \beta(\|f\|_{[-1, 1]}), \quad \beta : [0, \infty) \rightarrow [0, \gamma], \quad \beta \in C([0, \infty)) \text{ and nondecreasing}$$

$(\|f\|_{[a, b]} := \sup_{a \leq x \leq b} |f(x)|)$. We will later on remark on the extension of results to more realistic ionization rates.

The total current J is given by

$$(1.8) \quad J = J_n + J_p.$$

Note that J is a constant in $[-1, 1]$ because of (1.5).

The boundary conditions (at the Ohmic contacts) for (1.1)-(1.5) are

$$(1.8)(a) \quad np = \delta^4, \quad n - p - D = 0 \quad \text{at } x = \pm 1$$

where $\delta^2 (< 1)$ also originates from the scaling and

$$(1.8)(b) \quad \psi(1) = \ln \frac{n(1)}{\delta^2} - V, \quad \psi(-1) = \ln \frac{p(-1)}{\delta^2} + V.$$

$V \in \mathbb{R}$ is the (scaled) voltage applied to the diode (details on the scaling can be found in Markowich and Ringhofer (1982) and Markowich (1983)).

Because of our symmetry assumptions we restrain the investigation to 'symmetric' solutions, i.e. solutions which fulfill

$$(1.9) \quad \psi(x) = -\psi(-x), \quad n(x) = p(-x), \quad J_n(x) = J_p(-x), \quad x \in [-1, 1].$$

Another simplification is accomplished by employing the substitution

$$(1.10) \quad n = \delta^2 e^{\psi} u, \quad p = \delta^2 e^{-\psi} v, \quad J_n = \delta^2 e^{\psi} u', \quad J_p = -\delta^2 e^{-\psi} v'.$$

The system of equations obtained from (1.1)-(1.8) by using (1.9), (1.10) is

$$(1.11) \quad \lambda^2 \psi'' = \delta^2 e^{\psi} u - \delta^2 e^{-\psi} v - 1$$

$$(1.12) \quad (e^{\psi} u')' = -\alpha(\psi')(|e^{\psi} u'| + |e^{-\psi} v'|) \quad 0 \leq x \leq 1$$

$$(1.13) \quad (e^{-\psi} v')' = -\alpha(\psi')(|e^{\psi} u'| + |e^{-\psi} v'|)$$

subject to the boundary conditions

$$(1.14)(a) \quad \psi(0) = 0, \quad \psi(1) = \ln\left(\frac{1 + \sqrt{1 + 4\delta^4}}{2\delta^2}\right) - V$$

$$(1.14)(b) \quad u(0) = v(0), \quad u(1) = e^V$$

$$(1.14)(c) \quad v'(0) = -u'(0), \quad v(1) = e^{-V}.$$

The boundary conditions for u and v at $x = -1$ are $u(-1) = e^{-V}$, $v(-1) = e^V$. The maximum principle (see Protter and Weinberger (1967) applied to (1.12), (1.13) gives

$$(1.15) \quad u > e^{-|V|}, \quad v > e^{-|V|} \quad \text{on } [-1, 1].$$

Therefore n and p are positive (as physically required for densities). A solution

for $V = 0$ is given by $u \equiv 1$, $v \equiv 1$ and by solving

$$(1.16)(a) \quad \lambda^2 \phi'' = \delta^2 e^{\phi} - \delta^2 e^{-\phi} - 1, \quad 0 < x < 1$$

$$(1.16)(b) \quad \phi(0) = 0, \quad \phi(1) = \ln\left(\frac{1 + \sqrt{1 + 4\delta^4}}{2\delta^2}\right).$$

The solution $(V, \phi, u, v) = (0, \phi_e, 1, 1)$ where ϕ_e is the unique solution of (1.16) is called equilibrium solution. It implies $J = 0$ (the whole diode is in thermal equilibrium).

The two-point boundary-value problem (1.11)-(1.14) models the bias-controlled diode. In some cases it is more convenient to investigate the current-controlled device represented by the equations (1.11)-(1.13) subject to the boundary conditions.

$$(1.17)(a) \quad \phi(0) = 0, \quad \phi(1) = \ln\left(\frac{1 + \sqrt{1 + 4\delta^4}}{2\delta^2}\right) - \ln u(1)$$

$$(1.17)(b) \quad u(0) = v(0), \quad u(1)v(1) = 1, \quad u(1) > 0$$

$$(1.17)(c) \quad u'(0) = \frac{J}{2\delta^2}, \quad v'(0) = -\frac{J}{2\delta^2}.$$

(note that (1.17)(c) follows from

$$J = J_n(0) + J_p(0) = \delta^2 e^{\phi(0)} u'(0) - \delta^2 e^{-\phi(0)} v'(0) = 2\delta^2 u'(0) = -2\delta^2 v'(0)).$$

The problems (1.11)-(1.14) and (1.11)-(1.13), (1.17) are equivalent in the following sense. A solution (V_1, ϕ_1, u_1, v_1) of (1.11)-(1.14) yields the solution (J_1, ϕ_1, u_1, v_1) of (1.11)-(1.13), (1.17) where $J_1 = \delta^2 e^{\phi_1} u_1' - \delta^2 e^{-\phi_1} v_1'$ and a solution (J_2, ϕ_2, u_2, v_2) of (1.11)-(1.13), (1.17) yields the solution (V_2, ϕ_2, u_2, v_2) of (1.11)-(1.14) where $V_2 = \ln u_2(1)$.

There are numerous analytical and numerical investigations of the (even multi-dimensional) semiconductor device equations in the non-avalanche case (i.e. the recombination-generation rate R only depends on n and p) (see Mock (1983) for a rather

complete presentation of the results as well as for a collection of references). For the avalanche problem however there are (to the author's knowledge) only a few numerical studies (see Schütz (1982); Schütz, Selberherr and Pützl (1982)).

In this paper we regard (1.11)-(1.14) and (1.11)-(1.13), (1.17) as nonlinear eigenvalue problems (in the sense of Rabinowitz (1971), Krasnoselskii (1964) and investigate the solution set for nonpositive current

$$(1.18) \quad C^- = \{(I, \psi, u, v) \in (-\infty, 0] \times (C^2([0, 1]))^3 | (\psi, u, v) \text{ solves (1.11)-(1.13), (1.17) with } J = I\}$$

and the properties of the current-voltage ($J - V$) characteristic

$$(1.19) \quad J^- = \{(V, J) \in \mathbb{R} \times (-\infty, 0] \mid \text{there is } (\psi, u, v) \text{ such that } (J, \psi, u, v) \in C^- \text{ and } V = \ln u(1)\}.$$

The main theorem of this paper states that C^- contains an unbounded continuum (i.e. a closed and connected set in the $(-\infty, 0] \times (C^2([0, 1]))^3$ -topology) emanating from the equilibrium solution $(0, \psi_0, 1, 1)$ whose projection into $(-\infty, 0]$ equals $(-\infty, 0]$ (that means C^- contains solutions for all $J < 0$) and that the voltage $V \rightarrow -\infty$ as $J \rightarrow -\infty$. Therefore (1.11)-(1.14) has a solution (ψ, u, v) for every $V < 0$.

This result holds independently of the upper bound γ of the ionization rate α and carries over to more realistic α 's than given by (1.7). Therefore the conjecture that the branch of solutions of (1.11)-(1.14) breaks down if $\gamma > \frac{1}{2}$ holds (see Sze (1981)) is mathematically rejected at least for this model problem. We show, however, that the magnitude of γ has a decisive impact on the $J - V$ -characteristic. For $\alpha \equiv 0$ (nonavalanche case) the current fulfills $c_2 V < J < c_1 V$ for $V < 0$ while $|J|$ increases exponentially as $V \rightarrow -\infty$ for $\gamma > \frac{1}{2}$. ($c_1, c_2 > 0$ only depend on λ and δ). The exponential growth of the current represents the 'avalanche effect' and the diode 'breaks down' in real life when the current gets too large. We also show nonuniqueness for $V = 0$ for all α for which $\alpha(\psi_0)$ is sufficiently large.

The paper is organized as follows. Section 2 deals with the a priori estimates needed to prove existence of solutions for all $J < 0$ and Section 3 contains the existence proof and conclusions.

2. A PRIORI ESTIMATES

For the following we take $J < 0$.

At first we solve the continuity equations (1.12)-(1.13) for fixed $\psi \in C^1([0,1])$. We rewrite them as

$$\begin{aligned} (2.1)(a) \quad J'_n &= -\alpha(\psi')(|J_n| + |J_p|) \\ (2.1)(b) \quad J'_p &= \alpha(\psi')(|J_n| + |J_p|) \end{aligned} \quad \left. \vphantom{\begin{aligned} (2.1)(a) \\ (2.1)(b) \end{aligned}} \right\} \quad 0 \leq x \leq 1$$

with the initial values

$$(2.2)(c) \quad J_n(0) = \frac{J}{2}, \quad J_p(0) = \frac{J}{2}.$$

(2.1)(a) implies that J_n is nonincreasing; since $J_n(0) < 0$ we get $J_n < 0$ on $[0,1]$. $J_p(0) < 0$ holds and therefore we (initially) solve

$$J'_n = -\alpha|J|, \quad J'_p = \alpha|J|$$

(we mostly drop the argument ψ' of α) and get

$$(2.3)(a) \quad J_n = -\frac{|J|}{2}(2\alpha x + 1), \quad J_p = -\frac{|J|}{2}(-2\alpha x + 1) \quad \text{for} \quad \begin{cases} x \in [0,1] & \text{if } 0 < \alpha < \frac{1}{2} \\ x \in [0, \frac{1}{2\alpha}] & \text{if } \alpha > \frac{1}{2} \end{cases}$$

For $\alpha > \frac{1}{2}$ we have to solve

$$J'_n = \alpha(J_n - J_p), \quad J'_p = -\alpha(J_n - J_p), \quad \frac{1}{2\alpha} \leq x \leq 1$$

$$J_n\left(\frac{1}{2\alpha}\right) = -|J|, \quad J_p\left(\frac{1}{2\alpha}\right) = 0$$

and obtain

$$(2.3)(b) \quad J_n = -\frac{|J|}{2}(e^{2\alpha x-1} + 1), \quad J_p = -\frac{|J|}{2}(-e^{2\alpha x-1} + 1) \quad \text{for } x \in \left[\frac{1}{2\alpha}, 1\right].$$

u and v are computed from (1.10), (1.14)(b),(c):

$$(2.4)(a) \quad u = e^V + \frac{|J|}{2\delta^2} \cdot \begin{cases} \int_x^1 e^{-\psi(s)}(2as+1)ds & \text{for } x \in [0,1] \text{ if } 0 < a < \frac{1}{2} \\ \int_x^1 e^{-\psi(s)}(e^{2as-1}+1)ds & \text{for } x \in [\frac{1}{2a}, 1] \text{ if } a > \frac{1}{2} \\ \int_x^{1/2a} e^{-\psi(s)}(2as+1)ds + \int_{1/2a}^1 e^{-\psi(s)}(e^{2as-1}+1)ds & \text{for } x \in [0, \frac{1}{2a}] \text{ if } a > \frac{1}{2} \end{cases}$$

$$v = e^{-V} - \frac{|J|}{2\delta^2} \cdot \begin{cases} \int_x^1 e^{\psi(s)}(-2as+1)ds & \text{for } x \in [0,1] \text{ if } 0 < a < \frac{1}{2} \\ \int_x^1 e^{\psi(s)}(-e^{2as-1}+1)ds & \text{for } x \in [\frac{1}{2a}, 1] \text{ if } a > \frac{1}{2} \\ \int_x^{1/2a} e^{\psi(s)}(-2as+1)ds + \int_{1/2a}^1 e^{\psi(s)}(-e^{2as-1}+1)ds & \text{for } x \in [0, \frac{1}{2a}] \text{ if } a > \frac{1}{2} \end{cases}$$

We use the condition $u(0) = v(0)$ to relate V and J and get

$$(2.5) \quad V = \text{area} \sinh\left(\frac{J \cdot I(\psi)}{4\delta^2}\right), \quad J < 0$$

where the functional $I : C^1([0,1]) \rightarrow \mathbb{R}$ is given by

$$(2.6)(a) \quad I(\psi) = \int_0^1 [e^{\psi(s)}g_a(s) + e^{-\psi(s)}f_a(s)]ds$$

with

$$(2.6)(b) \quad g_a(x) = \begin{cases} -2ax+1 & \text{for } x \in [0,1] \text{ if } 0 < a < \frac{1}{2} \text{ and for } x \in [0, \frac{1}{2a}] \text{ if } a > \frac{1}{2} \\ -e^{2ax-1} + 1 & \text{for } x \in [\frac{1}{2a}, 1] \text{ if } a > \frac{1}{2} \end{cases}$$

$$(2.6)(b) \quad f_{\alpha}(x) = \begin{cases} 2\alpha x + 1 & \text{for } x \in [0, 1] \text{ if } 0 < \alpha < \frac{1}{2} \text{ and for } x \in [0, \frac{1}{2\alpha}] \text{ if } \alpha > \frac{1}{2} \\ e^{2\alpha x - 1} + 1 & \text{for } x \in [\frac{1}{2\alpha}, 1] \text{ if } \alpha > \frac{1}{2} \end{cases}$$

$I \in C^1([0, 1]) \rightarrow \mathbb{R}$ holds. For the estimates of the current J in terms of the voltage V we use

$$(2.7) \quad J = \frac{4\delta^2 \sinh V}{I(\psi)} \quad \text{if } I(\psi) \neq 0.$$

We collect the properties of u , v , J_n , J_p and J in:

Lemma 2.1: Assume that $0 < \alpha(\psi') < \frac{1}{2}$ holds. Then

- (i) $I(\psi) > (1 - 2\alpha) \int_0^1 e^{\psi(s)} ds + \int_0^1 e^{-\psi(s)} ds > 0$
- (ii) $J < 0 \iff V < 0, J = 0 \iff V = 0 \iff u \equiv v \equiv 1.$
- (iii) Let $V < 0$ hold.

Then $J_n < 0, J_p < 0$ on $[0, 1]$

- (iv) u is decreasing on $[0, 1], e^V < u < e^{-V}$
- (v) v is increasing on $[0, 1], e^V < v < e^{-V}.$

Lemma 2.2: Assume that $\alpha(\psi') > \frac{1}{2}$ is fixed. Then

- (i) there is a $\psi \in C^1([0, 1])$ such that $I(\psi) < 0$
- (ii) $J = 0 \implies (V = 0 \iff u \equiv v \equiv 1), J < 0 \iff (V < 0, I(\psi) > 0 \text{ or } V > 0, I(\psi) < 0)$
- (iii) $V = 0 \iff J = 0$ or $I(\psi) = 0.$
- (iv) Let $J < 0$ hold.

Then $J_n < 0$ on $[0, 1]; J_p < 0$ on $[0, \frac{1}{2\alpha}], J_p > 0$ on $(\frac{1}{2\alpha}, 1]$

- (v) u is decreasing on $[0, 1], u > e^V$
- (vi) v is increasing on $[0, \frac{1}{2\alpha}]$ and decreasing on $(\frac{1}{2\alpha}, 1], v > e^V.$

Therefore, for any solution $(V = 0, \psi^*, u^*, v^*) \neq (V = 0, \psi_e, 1, 1)$ of (1.11)-(1.14) (with $\gamma > \frac{1}{2}$) $I(\psi^*) = 0$ has to hold.

We now turn to Poisson's equation (1.11) subject to the boundary conditions

(1.14)(a). Differentiating (1.11) gives

$$(2.8)(a) \quad \lambda^2(\psi')'' = (n + p)\psi' + J, \quad 0 < x < 1$$

and (1.14) implies

$$(2.8)(b) \quad (\psi'(0))' = -\frac{1}{\lambda^2}, \quad (\psi'(1))' = 0.$$

The maximum principle (see Protter and Weinberger (1967)) yields

Lemma 2.3: $J < 0$ implies $\psi' > 0$ on $[0,1]$ and $V < \psi_0(1)$. Thus the a priori bounds

$$(2.9) \quad 0 < \psi(x) < \psi(1) = \psi_0(1) - V, \quad x \in [0,1]$$

follow from $J < 0$.

Differentiating (2.8)(a) and using (1.11)(a) yields

$$(2.10)(a) \quad \lambda^2(\psi'')' = [(n+p) + \lambda^2\psi'^2]\psi'' + (\psi')^2 + (J_n - J_p)\psi', \quad 0 < x < 1$$

$$(2.10)(b) \quad \psi''(0) = -\frac{1}{\lambda^2}, \quad \psi''(1) = 0.$$

From (2.3) we conclude that $J_n - J_p < 0$ in $[0,1]$ for $J < 0$. Since $\psi' > 0$ we obtain

Lemma 2.4. $J < 0$ implies $\psi'' > \frac{1}{\lambda^2}$ on $[0,1]$ and therefore

$$(2.11) \quad n(x) > p(x), \quad 0 < x < 1$$

holds.

Proof: $z = -\frac{1}{\lambda^2}$ is a lower solution of (2.10) and (2.11) follows from (1.1) (with $D \equiv 1$ on $[0,1]$).

We now derive upper bounds for n .

Lemma 2.5. Let $J < 0$ hold. Then

$$(2.12) \quad n(x) < \frac{1 + \sqrt{1 + 4\delta^4}}{2} + \frac{|J|}{2} \cdot \begin{cases} (2\alpha + 1)(1 - x) & \text{for } x \in [0,1] \text{ if } 0 < \alpha < \frac{1}{2} \\ (e^{2\alpha-1} + 1)(1 - x) & \text{for } x \in [\frac{1}{2\alpha}, 1] \text{ if } \alpha > \frac{1}{2} \\ \frac{1}{2} - 2x + (e^{2\alpha-1} + 1)(1 - \frac{1}{2\alpha}) & \text{for } x \in [0, \frac{1}{2\alpha}] \text{ if } \alpha > \frac{1}{2}. \end{cases}$$

Proof. We multiply (2.4)(a) by $\delta^2 e^\psi$ (getting n) and estimate $e^{\psi(x)-\psi(s)} < 1$ for

$s > x$ (ψ is nondecreasing). (2.12) is then obtained by integration and estimating

$$\frac{1}{2\alpha} (1 - e^{-2\alpha(1-x)}) < 1 - x.$$

In the case of zero generation ($\alpha \equiv 0$) we obtain upper bounds for n and p which are independent of J and V .

Lemma 2.6. Let $\alpha \equiv 0$ and $J < 0$ hold. Then $\psi'' < 0$ on $[0, 1]$ and

$$(2.13) \quad \rho(x) < n(x) < p(x) + 1, \quad 0 < x < 1$$

$$(2.14) \quad 0 < n(x) + p(x) < \sqrt{1 + 4\delta^4}, \quad 0 < x < 1$$

hold.

Proof. $\alpha = 0$ implies $J_n \equiv J_p \equiv \frac{J}{2}$ and therefore $\bar{z} = 0$ is an upper solution of (2.10). (2.13) follows from (1.1)(a). Also $(n + p)' = \psi'(n - p)$ holds. Thus $n + p$ is nondecreasing and $n + p < n(1) + p(1) = \sqrt{1 + 4\delta^4}$ follows.

We now derive lower and upper bounds for ψ using the estimates of n and p .

Lemma 2.7. Let $J < 0$ hold. Then

$$(2.15) \quad \psi(x) < \left(\psi(1) + \frac{1}{2\lambda^2}\right)x - \frac{x^2}{2\lambda^2}, \quad 0 < x < 1$$

follows. If $\alpha \equiv 0$

$$(2.16) \quad \psi(x) > -\frac{x^2}{2\lambda^2} + \psi(1)x$$

holds.

Proof. $\bar{\psi}(x) = \left(\psi(1) + \frac{1}{2\lambda^2}\right)x - \frac{x^2}{2\lambda^2}$ solves

$$\lambda^2 \bar{\psi}'' = -1, \quad \bar{\psi}(0) = 0, \quad \bar{\psi}(1) = \psi(1).$$

Lemma 2.4 implies that $\bar{\psi}$ is an upper solution of (1.11)(a). For $\alpha \equiv 0$ we obtain by integrating $\psi'' < 0$: $\psi(1) < \psi'(0)$ and from $\psi'' > -\frac{1}{\lambda^2}$

$$\psi(x) > -\frac{x^2}{2\lambda^2} + \psi'(0)x > -\frac{x^2}{2\lambda^2} + \psi(1)x.$$

To get a lower bound for ψ in the avalanche case we prove

Lemma 2.8. Let $J < 0$, $|J| > K_1$ hold. Also let $0 < \varepsilon < 1$ be such that $|J|\varepsilon^5 > K_2$ where K_1, K_2 only depend on λ, δ and γ . Then

$$(2.17) \quad \psi'(x) > \frac{1}{\varepsilon + q_\alpha(x)} + \sigma(x) \frac{1}{\sqrt{|J|\varepsilon^5}}, \quad 0 < x < 1$$

holds where $|\sigma|_{[0,1]} < K_3$ (K_3 only depends on λ, δ and γ) and

$$(2.18) \quad g_a(x) = \begin{cases} (2a+1)(1-x) & \text{for } x \in [0,1] \text{ if } 0 < a < \frac{1}{2} \\ (e^{2a-1} + 1)(1-x) & \text{for } x \in [\frac{1}{2a}, 1] \text{ if } a > \frac{1}{2} \\ \frac{1}{a} - 2x + (e^{2a-1} + 1)(1 - \frac{1}{2a}) & \text{for } x \in [0, \frac{1}{2a}] \text{ if } a > \frac{1}{2}. \end{cases}$$

Proof. The Lemmas 2.4 and 2.5 imply

$$\frac{n+p}{|J|} < \frac{2n}{|J|} < \frac{1 + \sqrt{1 + 4\delta^4}}{|J|} + g_a(x)$$

We now choose ε such that $1 + \sqrt{1 + 4\delta^4} < |J|\varepsilon$. Thus $\frac{n+p}{|J|} < \varepsilon + g_a(x)$ holds. Obviously the solution $y(\geq 0)$ of

$$(2.19)(a) \quad \frac{\lambda^2}{|J|} y'' = (\varepsilon + g_a(x))y - 1$$

$$(2.19)(b) \quad y'(0) = -\frac{1}{\lambda^2}, \quad y'(1) = 0$$

is a lower solution of (2.6), that means $0 < y < \psi^*$ holds on $[0,1]$. For large $|J|$ the problem constitutes a linear singularly perturbed (Neumann-type) boundary value problem with the reduced solution (obtained by setting $\frac{\lambda^2}{|J|}$ to zero)

$$y_r = \frac{1}{\varepsilon + g_a(x)}.$$

A standard singular perturbation analysis (see Howes (1978)) which takes the possible smallness of ε into account ($y_r(1) = \frac{1}{\varepsilon}$!) gives

$$\|y - y_r\|_{[0,1]} < K_3 \frac{1}{\sqrt{|J|\varepsilon^5}}$$

whenever $|J| > K_1$, $|J|\varepsilon^5 > K_2$. This implies (2.17).

A lower bound for ψ follows by integrating (2.17):

Lemma 2.9. Let the assumption of Lemma 2.7 hold. Then

$$(2.20) \quad \psi(x) > \nu(x) \frac{1}{\sqrt{|J|\epsilon^5}} + \begin{cases} L_1 + \frac{1}{2\alpha+1} \ln \left(\frac{1}{\epsilon + (2\alpha+1)(1-x)} \right) \\ \text{for } x \in [0, 1] \text{ if } 0 < \alpha < \frac{1}{2} \\ L_2 \text{ for } x \in [0, \frac{1}{2\alpha}] \text{ if } \alpha > \frac{1}{2} \\ L_3 + \frac{1}{e^{2\alpha-1} + 1} \ln \left(\frac{1}{(e^{2\alpha-1} + 1)(1-x) + \epsilon} \right) \\ \text{for } x \in [\frac{1}{2\alpha}, 1] \text{ if } \alpha > \frac{1}{2} \end{cases}$$

holds where $\|\psi\|_{[0,1]} \leq K_3$ and L_1, L_2, L_3 only depend on γ .

Since ϵ can be made arbitrarily small when $J \rightarrow -\infty$ (still keeping $|J|\epsilon^5$ large) the Lemmas 2.8 and 2.9 imply that $\|\psi'\|_{[0,1]} (> \psi'(1) > K_4 \frac{1}{\epsilon})$ and $\psi(1) (> K_5 \ln \frac{1}{\epsilon})$ become unbounded as $J \rightarrow -\infty$.

We also need an upper bound of ψ' :

Lemma 2.10. $J < 0$ implies

$$(2.21) \quad (0 <) \psi' < K_6 e^{|V|} (|J| + 1)$$

where K_6 only depends on λ and δ .

Proof: (2.4)(a) implies $n + p > n > \delta^2 e^V$. Thus the solution w of

$$\lambda^2 w'' = \delta^2 e^V w - |J|, \quad w'(0) = -\frac{1}{\lambda^2}, \quad w'(1) = 0$$

is an upper solution of (2.8). Therefore

$$\psi'(x) < w(x) = e^{-V} \frac{|J|}{\delta^2} + \frac{\cosh(\frac{\delta}{\lambda} e^{V/2} (1-x))}{\lambda \delta e^{V/2} \sinh(\frac{\delta}{\lambda} e^{V/2})}$$

and (2.21) follows since $V < \psi_0(1)$.

We now employ the derived bounds to get a priori estimates on the current-voltage characteristic.

Theorem 2.1. Assume that $(J, \psi, u, v) \in C^-$ and $(V, J) \in J^-$. Then

$$(2.22) \quad c_1 |V| < |J| < c_2 |V| \text{ if } \alpha(\psi') = 0$$

$$(2.23) \quad C_1 |V| < |J| < C_3 e^{|V|} (1 - e^{-2|V|}) \quad \text{if } 0 < \alpha(\psi') < \frac{1}{2}$$

holds. If $\alpha(\psi') > \frac{1}{2}$ then

$$(2.24) \quad C_4 (1 - e^{-2|V|}) e^{|V|} (1 - 1/2\alpha) < |J| < D \exp((5 + \mu)(e^{2\alpha-1} + 1)|V|)$$

holds for $\frac{1}{2} < \alpha(\psi') < \alpha_0$ with some $\alpha_0 < 1$ and every $\mu > 0$. If $\alpha(\psi') > \alpha_0$ then

$$(2.25) \quad C_5 (1 - e^{-2|V|}) e^{|V|} < |J| < D \exp((5 + \mu)(e^{2\alpha-1} + 1)|V|).$$

The constants C_1, \dots, C_5 only depend on λ, δ and γ . D depends on λ, δ, γ and μ .

Proof. $0 < \alpha < \frac{1}{2}$ and $\psi > 0$ imply $0 < I(\psi) < 2 \int_0^1 \cosh \psi(s) ds$ and (2.15), (2.7) yield:

$$0 < I(\psi) < \frac{2 \sinh(\psi(1) + \frac{1}{2\lambda^2})}{\psi(1) + \frac{1}{2\lambda^2}}.$$

Thus

$$|J| > \frac{\delta^2 (1 - e^{-2|V|}) (\psi(1) + \frac{1}{2\lambda^2})}{e^{-|V|} \sinh(\psi(1) + \frac{1}{2\lambda^2})}.$$

and the lower bounds for $|J|$ in (2.22), (2.23) follow. $\alpha \geq 0$ implies

$$I(\psi) = 2 \int_0^1 \cosh \psi(s) ds \quad \text{and the estimate (2.16) gives}$$

$$I(\psi) > 2 \frac{\sinh(\psi(1) - \frac{1}{2\lambda^2})}{\psi(1) - \frac{1}{2\lambda^2}}$$

when $|V|$ is so large that $\psi(1) - \frac{1}{2\lambda^2} > 0$. We derive (using (2.7))

$$|J| < \frac{\delta^2 (1 - e^{-2|V|}) (\psi(1) - \frac{1}{2\lambda^2})}{e^{-|V|} \sinh(\psi(1) - \frac{1}{2\lambda^2})}$$

and (2.22) is proven. For $0 < \alpha < \frac{1}{2}$ we estimate

$$I(\psi) > \int_0^1 (-2\alpha s + 1) ds = 1 - \alpha.$$

Thus

$$|J| < \frac{2\delta^2 e^{|\psi|} (1 - e^{-2|\psi|})}{1 - \alpha}$$

and the upper bound in (2.23) follows. Now let $\alpha(\psi') > \frac{1}{2}$. Then, since ψ is nondecreasing and since $g_\alpha(x)$ is positive in $[0, \frac{1}{2\alpha}]$ and negative in $[\frac{1}{2\alpha}, 1]$:

$$\begin{aligned} I(\psi) &< \int_0^1 f_\alpha(s) ds + e^{\psi(1/2\alpha)} \int_0^1 g_\alpha(s) ds = \\ &= 1 - \frac{1}{4\alpha} + \frac{1}{2\alpha} e^{2\alpha-1} + e^{\psi(1/2\alpha)} \left(1 + \frac{1}{4\alpha} - \frac{1}{2\alpha} e^{2\alpha-1} \right) \end{aligned}$$

($f_\alpha > 0$ in $[0, 1]$) holds. The function $h(\alpha) = 1 + \frac{1}{4\alpha} - \frac{1}{2\alpha} e^{2\alpha-1}$ has a unique zero $\alpha_0 \in (\frac{1}{2}, 1)$. Thus

$$I(\psi) < \begin{cases} 2, & \alpha > \alpha_0 \\ 1 - \frac{1}{4\alpha} + \frac{1}{2\alpha} e^{2\alpha-1} + \frac{1}{2} e^{\psi(1/2\alpha)}, & \frac{1}{2} < \alpha < \alpha_0 \end{cases}$$

holds. If $I(\psi) > 0$ the lower bound in (2.25) follows and the lower bound in (2.24) (also for $I(\psi) > 0$) is implied by (2.15) which gives

$$\psi\left(\frac{1}{2\alpha}\right) < c + |\psi| \frac{1}{2\alpha}.$$

Lemma 2.9 implies that ' $V \rightarrow -\infty$ as $J \rightarrow -\infty$ ' and therefore $I(\psi) < 0$ can (for $J < 0$) only hold for $|J| < F$, where F depends on λ, δ and γ . This proves the lower bounds in (2.24), (2.25).

(2.20) yields $\psi(1) = \psi_0(1) - V > L_3 + \frac{1}{e^{2\alpha-1} + 1} \ln \frac{1}{c}$. We set $\varepsilon = 1/|J|^{1/5-\sigma}$ for $0 < \sigma < \frac{1}{5}$ and obtain the upper bounds in (2.24), (2.25) for $|J|$ suff. large since $V < \psi_0(1)$ holds.

Since Lemma 2.8 implies that $\alpha(\psi') \rightarrow \gamma$ as $J \rightarrow -\infty$ (or as $V \rightarrow -\infty$). The theorem proves that the current J increases (in absolute value) at least (and at most) exponentially as $V \rightarrow -\infty$ in the avalanche case $\gamma > \frac{1}{2}$. For zero generation ($\gamma = 0$) the

increase is at most (and at least) linear. For the intermediate case $0 < \gamma < \frac{1}{2}$ the increase is at least linear and at most exponential. The author conjectures that the distinction of the cases $\frac{1}{2} < \gamma < \alpha_0$ and $\gamma > \alpha_0$ only comes in for technical reasons and that (2.25) holds for all $\gamma > \frac{1}{2}$.

3. EXISTENCE THEOREMS

We need the following

Lemma 3.1: Let w be the (unique) solution of the problem

$$(3.1)(a) \quad w'' = a(x)e^{w+\eta} - b(x)d^{-w-\eta} + f(x), \quad 0 < x < 1$$

$$(3.1)(b) \quad w(0) = \mu_0, \quad w(1) = \mu_1$$

where $a, b \in C([0,1])$; $a, b > 0$ on $[0,1]$; $\eta \in \mathbb{R}$. Then $w = w(\eta, \mu_0, \mu_1, f, \cdot)$ regarded as mapping from $\mathbb{R}^3 \times C^1([0,1])$ into $C^1([0,1])$ is completely continuous.

Proof: We take $\eta \in [\bar{\eta}, \bar{\eta}]$, $\mu_0 \in [\underline{\mu}_0, \bar{\mu}_0]$, $\mu_1 \in [\underline{\mu}_1, \bar{\mu}_1]$ and denote

$$\underline{a} := \min_{0 \leq x \leq 1} a(x), \quad \bar{a} := \max_{0 \leq x \leq 1} a(x) \quad (\text{analogously for } b). \quad \text{Then the unique solution } w_1 \text{ of}$$

$$w_1'' = \underline{a}e^{w_1+\eta} - \bar{b}e^{-w_1-\eta} + f|_{[0,1]}, \quad 0 < x < 1$$

$$w_1(0) = \underline{\mu}_0, \quad w_1(1) = \underline{\mu}_1$$

is a lower solution of (3.1) and the unique solution w_2 of

$$w_2'' = \bar{a}e^{w_2+\eta} - \underline{b}e^{-w_2-\eta} - f|_{[0,1]}, \quad 0 < x < 1$$

$$w_2(0) = \bar{\mu}_0, \quad w_2(1) = \bar{\mu}_1$$

is an upper solution of (3.1), i.e. $w_1 < w < w_2$ on $[0,1]$ holds. Since

$$|w''|_{[0,1]} < \bar{a}e^{w_2+\eta} + \bar{b}e^{-w_1-\eta} + |f|_{[0,1]}$$

holds, Ascoli's theorem implies that $w : \mathbb{R}^3 \times C^1([0,1]) \rightarrow C^1([0,1])$ maps bounded sets into precompact sets. The continuity of w is immediate.

Now we prove the basic

Theorem 3.1: For any $\gamma > 0$ the solution set C^- of (1.11)-(1.13), (1.17) contains an unbounded continuum \tilde{C}^- (in the $(-\infty, 0] \times (C^1([0, 1]))^3$ -topology) emanating from the equilibrium solution $(0, \phi_0, 1, 1)$ whose projection into $(-\infty, 0]$ equals $(-\infty, 0]$ (i.e. \tilde{C}^- contains a solution (J, ϕ, u, v) for every $J < 0$).

Proof: We regard $V = V(J, \phi)$ (given by (2.5)) as functional $V : (-\infty, 0] \times C^1([0, 1]) \rightarrow \mathbb{R}$. The continuity of I implies the continuity of V . Using (2.3)-(2.5) we rewrite Poisson's equation (1.11) as

$$(3.2)(a) \quad \lambda^2 \phi'' = \delta^2 e^{\phi+V(J, \phi)} - \delta^2 e^{-\phi-V(J, \phi)} - 1 + |J|G(\phi)(x), \quad 0 < x < 1$$

$$(3.2)(b) \quad \phi(0) = 0, \quad \phi(1) = \phi_0(1) - V(J, \phi)$$

with

$$(3.3) \quad G(\phi)(x) = e^{\phi(x)} \int_x^1 e^{-\phi(s)} \left| \frac{J(s)}{J} \right| ds + e^{-\phi(x)} \int_x^1 e^{\phi(s)} \left| \frac{J(s)}{J} \right| ds$$

(note that $\left| \frac{J(x)}{J} \right|, \left| \frac{J(x)}{J} \right|$ are independent of J , they only depend on $\alpha(\phi')$ and on x). $G : C^1([0, 1]) \rightarrow C^1([0, 1])$ is continuous since $\alpha : C^1([0, 1]) \rightarrow [0, \gamma]$ is continuous.

We set $\psi = \phi_0 + \phi$ and rewrite (3.2) as the fixed point problem $\phi = T(J, \phi)$ where $y = T(J, z)$ is defined as the unique solution of the problem

$$(3.4)(a) \quad \lambda^2 y'' = \delta^2 \exp(y + \phi_0 + V(J, \phi_0 + z)) - \delta^2 \exp(-y - \phi_0 - V(J, \phi_0 + z)) - 1 - \lambda^2 \phi_0'' + |J|G(\phi_0 + z), \quad 0 < x < 1$$

$$(3.4)(b) \quad y(0) = 0, \quad y(1) = -V(J, \phi_0 + z).$$

Lemma 3.1 and the continuity of V and G imply that $T : (-\infty, 0] \times C^1([0, 1])$ is completely continuous. $V(0, w) \geq 0$ and therefore $y = T(0, z)$ is given by the solution of

$$(3.5)(a) \quad \lambda^2 y'' = \delta^2 e^{y+\phi_0} - \delta^2 e^{-y-\phi_0} - 1 - \lambda^2 \phi_0'', \quad 0 < x < 1$$

$$(3.5)(b) \quad y(0) = y(1) = 0.$$

$y \geq 0$ follows. From Rabinowitz (1971, Theorem 3.2) we conclude that the solution set of $T(J, \rho) = \phi$ contains an unbounded continuum \tilde{E}^- (in the $(-\infty, 0] \times C^1([0, 1])$ -topology).

Theorem 2.1 and Lemma 2.3 imply

$$|V| \leq \max(\phi_0(1), \frac{|J|}{C_1})$$

and

$$-\phi_0 < \phi < \phi_0(1) - \phi_0 + |V|.$$

Lemma 2.10 yields

$$-\phi_0' < \phi' < K_0 e^{|V|} (|J| + 1) - \phi_0'.$$

We conclude from these estimates

$$\|\phi\|_{[0,1]} + \|\phi'\|_{[0,1]} \leq M(1 + |J| + e^{|J|/C_1} + e^{|J|/C_1}|J|)$$

where $M > 0$ is independent of J (and V). Since E^- is unbounded it has to contain solutions (J, ϕ) for all $J \leq 0$. The statement of the theorem follows by observing that u and v as given by (2.4) are continuous as functions of (J, ϕ) in the $(-\infty, 0] \times C^1([0,1])$ -topology.

The most important implication of Theorem 3.1 is:

Theorem 3.2: For any $\gamma > 0$ the current-voltage characteristic J^- contains a continuous curve Γ^- emanating from $(0,0)$ whose projection Γ_J^- into the J -axis equals $(-\infty, 0]$ and whose projection Γ_V^- into the V -axis fulfills

$$(3.6)(a) \quad \Gamma_V^- = (-\infty, 0] \quad \text{for } 0 < \gamma < \frac{1}{2}$$

$$(3.6)(b) \quad (-\infty, 0] \subseteq \Gamma_V^- \subseteq (-\infty, \phi_0(1)] \quad \text{for } \gamma > \frac{1}{2}.$$

Proof: Theorem 3.1 implies that J^- contains a continuum O^- emanating from $(0,0)$ whose projection into the J -axis equals $(-\infty, 0]$. From Lemma 2.9 we conclude that

$\phi(1) = \phi_0(1) - V$ is positive and unbounded as $J \rightarrow -\infty$. Lemma 2.1, (ii) implies that $V < 0$ for $J < 0$ and $0 < \gamma < \frac{1}{2}$. Therefore (3.6)(a) follows. (3.6)(b) is concluded by noting that $\phi(1) = \phi_0(1) - V > 0$ for $J < 0$ holds (see Lemma 2.3).

The obvious consequence for the solution set

$$(3.7) \quad D^- = \{(U, \phi, u, v) \in (-\infty, 0] \times (C^2([0,1]))^3 \mid (\phi, u, v) \text{ solves (1.11)-(1.14) for } V = U\}$$

of the voltage-controlled diode is

Corollary 3.1 D^- contains an unbounded continuum \tilde{D}^- containing $(0, \psi^*, v^*, u^*)$ whose projection into $(-\infty, 0]$ equals $(-\infty, 0] \cdot (\psi^*, u^*, v^*) = (\psi_0, 1, 1)$ if the equilibrium solution is unique (e.g. for $0 < \gamma < \frac{1}{2}$). $J < 0$ holds for every $(V, \psi, u, v) \in \tilde{D}^-$. (\tilde{D}^- contains a solution for every $V < 0$).

We now show that multiple solutions of (1.11)-(1.14) for $V = 0$ can occur.

Theorem 3.3: Assume that $\alpha(\psi_0') > \frac{1}{2}$ where $\frac{1}{2}$ only depends on λ and δ . Then there is a solution (ψ^*, u^*, v^*) of (1.11)-(1.14) for $V = 0$ which is different from the equilibrium solution $(\psi_0, 1, 1)$ and $J^* = \delta^2 \alpha^{\psi^*}(u^*)' - \delta^2 \alpha^{-\psi^*}(v^*)' < 0$ holds.

Proof: We obtain from (2.6)

$$I(\psi) = 2 \left(\int_0^1 \cosh \psi(s) ds - \int_0^1 h_\alpha(s) \sinh \psi(s) ds \right)$$

where

$$h_\alpha(x) = \begin{cases} 2\alpha x, & 0 \leq x \leq \frac{1}{2\alpha} \\ e^{2\alpha x - 1}, & \frac{1}{2\alpha} < x < 1 \end{cases} \quad (\text{for } \alpha(\psi_0') > \frac{1}{2}).$$

Obviously $h_\alpha(x) > 2\alpha x$ on $[0, 1]$ and therefore

$$I(\psi) < 2 \left(\int_0^1 \cosh \psi(s) ds - 2\alpha \int_0^1 s \sinh \psi(s) ds \right)$$

holds. Choosing α such that

$$\alpha(\psi_0') > \frac{1}{2} \frac{\int_0^1 \cosh \psi_0(s) ds}{\int_0^1 s \sinh \psi_0(s) ds}$$

implies $I(\psi_0) < 0$. Thus there is a neighbourhood N of 0 in $C^1([0, 1])$ with

$I(\psi_0 + \phi) < 0$ for $\phi \in N$. (2.5) implies $V(J, \psi_0 + \phi) > 0$ for $(J, \phi) \in (-\infty, 0) \times N$.

Since the continuum E^- (used in the proof of Theorem 3.1) emanates in $(0, 0)$. The intersection of E^- and $(-\infty, 0) \times N$ is not empty. This implies that there are solutions of (1.11)-(1.13), (1.17) for $J < 0$ for which $V > 0$ holds. Since V is negative for

$J < 0$, $|J|$ sufficiently large, there has to be $(J, \phi^*) \in E^-$, $J \neq 0$, with

$I(\phi_0 + \phi^*) = 0$. This gives $V(J, \phi_0 + \phi^*) = 0$ and Theorem 3.3 follows.

The condition $I(\phi_0) < 0$ implies that J^- is not contained in $(-\infty, 0)^2$ (see Figure 2).

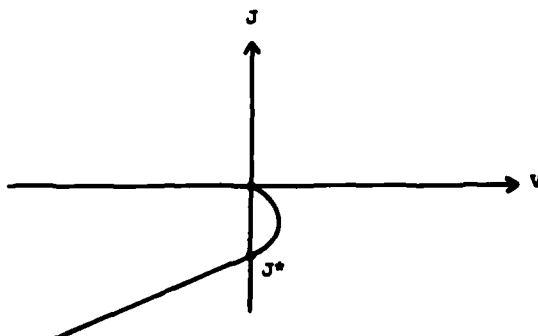


Figure 2. Qualitative structure of the $J - V$ -characteristic for $I(\phi_0) < 0$.

However, $I(\phi_0) < 0$ is physically unreasonable and it is not clear whether the nonuniqueness of the equilibrium solution prevails if $I(\phi_0) > 0$.

We conclude from the Theorems 2.1 and 3.2 that the avalanche case $\gamma > \frac{1}{2}$ is distinguished from the nonavalanche case $0 < \gamma < \frac{1}{2}$ by a more rapid decrease of the current J as $V \rightarrow -\infty$ (see Figure 3).

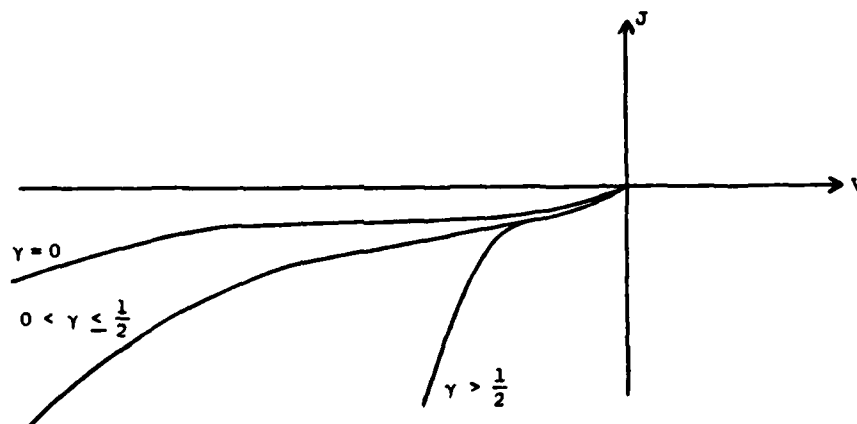


Figure 3. Qualitative Structure of the $J - V$ -characteristic for various γ 's ($I(\phi_0) > 0$ is assumed) for reverse bias.

We remark that an investigation of (1.11)-(1.13), (1.17) for nonnegative current can be done in a similar fashion. A relation of the term (2.5) (with a different functional I) holds and the existence of an unbounded continuum of solutions E^+ follows as in the proof of Theorem 3.1. To conclude that E^+ contains solutions for all $J > 0$ additional a priori estimates have to be obtained (since the estimates in Section 2 only hold for $J < 0$). For the case $\alpha \equiv 0$ these estimates are given in Mock (1983), Markowich (1983).

Finally, assume that α is not a functional on $C^1([0,1])$ but simply a function (3.8) $\alpha : \mathbb{R} \rightarrow [0, \gamma]$, α is continuous and nondecreasing, such that the ionization rate $\alpha(\psi'(x))$ is space-dependent. Then (2.4)-(2.6) have to be modified by substituting ' α ' in the integrands by $\int_0^s \alpha(\psi'(s))ds$ and ' $\frac{1}{2\alpha}$ ' in the integration intervals by that value $\bar{x} \in [0,1]$ for which $\int_0^{\bar{x}} \alpha(\psi'(s))ds = \frac{1}{2}$ holds. Theorem 3.1 still holds. By estimating (2.12) in terms of γ an analogue of (2.18), (2.20) (also in terms of γ) implying ' $\psi(1) \rightarrow \infty$ as $J \rightarrow \infty$ ' is obtained and Theorem 3.2 and Corollary 3.1 follow. The estimates of the current-voltage characteristic given in Theorem 2.1 for $0 < \gamma < \frac{1}{2}$ still hold. The avalanche-estimate (2.25) (with α in the upper bound substituted by γ) holds if for example $\alpha(\tau) > \sigma_1 e^{-\sigma_2/|\tau|}$ with σ_1 suff. large and σ_2 suff. small.

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Nonlinear eigenvalue problems, unbounded solution continua, two-point boundary-value problems, impact ionization, semiconductor		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper is concerned with the analysis of the solution set of the two- point boundary value problem modelling the avalanche effect in semiconductor diodes for negative applied voltage. This effect is represented by a large increase of the absolute value of the current starting at a certain reverse basis. We interpret the avalanche-model as a nonlinear eigenvalue problem (cont.)		

ABSTRACT (cont.)

(with the current as eigenparameter) and show (using a priori estimates and a well known theorem on the structure of solution sets of nonlinear eigenvalue problems for compact operators) that there exists an unbounded continuum of solutions which contains a solution corresponding to every negative voltage. Therefore, the solution branch does not "break down" at a certain threshold voltage (as expected on physical grounds). We discuss the current-voltage characteristic and prove that the absolute value of the current increases at most (and at least) exponentially in the avalanche case as the voltage decreases to minus infinity.

84